Investigation of the Dynamic Properties of Railway Tracks using a Model for Calculation of Generation of Wheel/Rail Noise

Hyo-In Koh† and Anders Nordborg*

Abstract

For optimization of a low-noise track system, rail vibration and noise radiation needs to be investigated. The main influencing parameters for the noise radiation and the quantitative results of every track system can be obtained using a calculation model of generation and radiation of railway noise. This kind of model includes contact modeling and the calculation model of the dynamic properties of the wheel and the rail. This study used a nonlinear wheel/rail interaction model in the time domain to investigate the excitation of the rolling noise. Wheel/rail response is determined by time integrating Green's function of the rail together with force impulses from the wheel/rail contact. This model and the results of the study can be used for supporting calculation with the conventional model by an addition of the contributions due to nonlinearities to the roughness spectrum.

Keywords: Wheel/Rail noise, Rail vibration, Rail noise, Roughness spectrum

1. Introduction

1.1 Rolling noise model

In the most conventional calculation models of noise generation and radiation from wheel/rail rolling in railway systems, the contact effects are assumed to be linear. The models are time invariant and the calculation is performed in the frequency domain [1–7]. A nonlinear wheel/rail interaction model in the time domain can be used if it is necessary to consider nonlinearities in the noise excitation. The deflections of the wheel and the rail are determined by time integrating Green’s function of the rail combined with force impulses from the wheel/rail contact. Green’s function of the rail includes discrete supports, thus accounting for the sleeper-passing frequency and the pinned-pinned frequency, which causes an additional parametric excitation. This model is also useful because vibration propagation and decay along the track are properties of Green’s function of the periodically supported rail. The model is explained more in detail in [16]; this paper is primarily aimed at investigating the model by means of calculation examples of rail/wheel contact force and rail vibration for different track types including rails in Korea.

1.2 Theory of the model

1.2.1 Rail Green’s function in the frequency domain

Green’s function, \( G_\omega(x, x_0) \), in the frequency domain for a periodically supported rail is derived in detail in [8, 9, 10, 11]. This is needed to determine the interactive contact force of the rail/wheel and the vibration levels and the radiated noise levels of the rail. The main parameters are described in Fig. 1.

![Fig. 1 Discretely supported rail, excited by a unit point force](image-url)
1.2.2 Point force excitation

Sound power radiation from a vibration surface

\[ W = \frac{1}{2} \rho_0 c_0 G S \langle v^2 \rangle \] (1)

where \( \rho_0 \) is the density of air, \( c_0 \) is the air speed, \( \sigma \) is the radiation efficiency, and \( S \) is the area of the radiating surface. \( \langle v^2 \rangle \) is the mean square of the vibrating velocity of the radiating surface.

Consider a unit point force excited infinite rail. The vibration amplitude decays away from the force point. The total radiation from the infinite rail is

\[ W = \frac{1}{2} \rho_0 c_0 w \int_{-\infty}^{\infty} |G(x|x_0)|^2 dx \] (2)

where \( w \) is the width of the rail, which may be rewritten

\[ W = \frac{1}{2} \rho_0 c_0 \sigma w \int_{-\infty}^{\infty} G(x|x_0)|^2 dx \]

(3)

where \( \omega \) is the circular frequency. \( G(x|x_0) \) is Green’s function of the periodically, sleeper-supported, infinite rail [8-10]. The force acts in the point \( x_0 \), and \( x \) is the response position. The decay of radiation vibrations along the rail in the \( x \)-direction is automatically taken into account by the propagation coefficients in Green’s function.

1.2.3 Rail/wheel contact force excitation

The contact force \( F_0 \) between the rail and wheel excites the rail. The expression for sound radiation becomes

\[ W = \frac{1}{2} \rho_0 c_0 \sigma w \int_{-\infty}^{\infty} |G(x|x_0)|^2 dx \]

(4)

The contact force depends on rail/wheel running surface roughness levels, geometrical filtering and compression in the contact region, which are nonlinear processes, and rail and wheel receptances, as well as variation in the rail receptance along the rail with wheel position due to the discrete supports. It can be calculated, according to section 1.2.4, in the time-domain with the rail/wheel simulation program [11], and the nonlinear state-dependent rolling contact model [12].

1.2.4 Contact force calculation

A wheel with mass \( M_w \) and preload \( P \) rolls with forward velocity \( v \) over a rail (Fig. 2). The rail has mass \( M_r \) per unit length, bending stiffness \( E I \). In the figure, the rail symbolically has a corrugation \( r_0 \sin(2\pi/\lambda_0)x \) on the running surface. The coordinate axis for \( x \) points in the forward running direction, and \( y \) upwards in the vertical direction. A perfectly smooth rail implies that \( y_r = 0 \) \( \mu \)m. The rail is periodically supported at discrete points with spacings \( L \). Each support consists of a spring-mass-spring combination: \( K_p \) denotes pad stiffness with loss factor \( \eta_p \), \( M_s \) is the sleeper mass and \( K_b \) means the ballast stiffness with loss factor \( \eta_b \). The compression in the rail-wheel contact region \( \delta_{\text{lim}} = y_r + y_s - y_w \) is a function of vertical deflection of the rail \( y_r \) and of the wheel, \( y_w \) plus the deviation from a perfect running surface \( y_c \). The contact force \( f \), caused by the compression, excites the wheel (upwards) and the rail (downwards). Since this is a timedomain model, rail and wheel deflections plus the contact force are calculated at each discrete point \( x_0 = nL \times \Delta x \), where \( n \) is an integer ad \( \Delta x \) the space increment, with the corresponding time increment \( \Delta t \times \sqrt{v} \).

The vertical rail deflection, \( y_r(x_0, t) \) at the moving point, \( x_0 = vt \), under the wheel is a convolution of the rail’s Green’s function, \( g(x, x_0; t, t_0) \), and the contact force \( f(x, t) \),

\[ y_r(x_0,t_0) = \int \int g(x,x_0; t_0)(x,t)dxdt \] (5)

After discretization, and considering that the contact force moves forward,

\[ f(x, t) = f(t) \delta(x-\Delta x), \] (6)

where \( k \) is an integer. Note that the force point, \( x = x_0 - k\Delta x \), is expressed relative to the response point, \( x_0 \). Previous calculations provided Green’s function in the frequency domain, \( G_o(x, x_0) \), here first made conjugate-symmetric \( \tilde{G}_o = G_o^* \). Here \( \tilde{G}_o \) is the complex conjugate, to ensure reality and causality, which is then trans-
formed to the time domain by a discrete Fourier transform, 
\[ g(x, y, t, f_0) = \text{DFT}^{-1} [G_g(x, y)] \]. The force, \( f(t_0-k\Delta t) \), known in the past where \( k>0 \), must be found by iterative improvements for the very last time step, \( k = 0 \). The total number of points, \( K \), must be so great that the impulse from the most distant point, \( x_0-(K-1)\Delta x \), has decayed to a negligible amplitude before it arrives at the response point, \( x_0 \).

The wheel model is the simplest possible: a rigid mass, \( M_w \), acted upon vertically by a constant preload force, \( P \), plus the contact force, \( f \). Numerically, e.g. with Runge-Kutta, solving the ordinary differential equation.

\[ M_w \ddot{y}_W(t) = f(t) - P \]

The compression of the wheel/rail contact region

\[ \delta_{\text{lim}} = y_x - y_w \]  

is a function of the contact force. Being unknown at the current time step, it can be approximated first with its most recent value, \( f(t_0) \approx f(t_0 - \Delta t) \). According to Hertz, the compression distance is a nonlinear function of the contact force, \( f \),

\[ \delta_H = \left[ \frac{2(1 - \nu)}{G R_a} \right]^{\frac{1}{2}} \alpha_0 \]  

\[ 0 < \delta_H(f) < \delta_{\text{lim}} \]

iteratively yields the Hertzian contact force, \( f \). It is then possible to iterate, with the successive force improvements \( f(t_0) \approx f \), until the error \( \int |f(t_0) - f| < \varepsilon \), which is a small number. The contact force \( f \) plus the deflection of the rail and the wheel, \( y_x \) and \( y_w \), are now known at the current time step.

1.2.5. State-dependent contact model

The compression \( \delta \) in the contact between wheel and rail must fulfill the geometric relation

\[ \delta = y - y_w + y_x \]  

where \( y_x \) is the surface texture height. In an iterative manner for each time-step, the wheel and rail displacements as well as the contact force are determined by balancing the geometric relation stated above with the state-dependent contact force expression

\[ f(\delta, r) = \begin{cases} k_r(\delta)(r - y_x(r)), & \delta > 0 \\ 0, & \delta \leq 0 \end{cases} \]  

where \( k_r(\delta) \) is the state-dependent contact stiffness and \( y_x(r) \) are the state-dependent line texture heights.

Definitions and methodology of the state-dependent force formulation, which here is applied to the wheel-rail contact, was previously presented in [12]. In summary, the widely used Hertzian contact stiffness and the linearized contact filter are replaced by state-dependent contact stiffness and contact filters which vary with the relative displacement of the wheel and the rail. Prior to the time-domain dynamic model of the of the wheel and the rail, the state-dependent contact stiffness as well as state-dependent contact filtering, which is in the form of filtered texture heights, are pre-calculated using quasi-static modeling schemes. For this, Boussinesq contact modeling theory is employed, which includes the three-dimensional topography of the wheel and rail within the contact area.

1.2.6. Contact force calculation in the frequency domain

It is shown in [8], in a way similar to the TWINS simulation program, that a linear approximation for the contact force spectrum \( S_{F,F_0} \) between rail and wheel can be written

\[ S_{F,F_0} = \frac{1}{\nu} \Phi_{\omega_0}(k_0) \Phi_{\omega_0}(k_0) \left| H(k_0) \right|^2 \]

where \( \nu \) is the train speed, \( \Phi_{\omega_0}(k_0) \) is the roughness spectrum of the running surface of the rail, \( \Phi_{\omega_0}(k_0) \) is the roughness spectrum of the running surface of the wheel, and \( \left| H(k_0) \right|^2 \) is the contact filter. \( k_0 \) means the linearized Hertzian contact spring stiffness, \( \alpha_0 \), is the vertical rail receptance, and \( \alpha_w \) is the radial receptance of the wheel, \( k_0 = \frac{2\pi}{\lambda_0} \) is the wavenumber of the surface roughness, where \( \lambda_0 \) is the wavelength of the surface roughness.

2. Calculation of the Dynamic Parameters of the Rail

2.1 Input data

The properties of the rail vibration, such as the rail receptance, propagation of the vibration along the rail and the contact force at the rail/wheel interaction, are calculated for a few different types of railway lines with different track parameters and train speeds. Different modeling methods are also used to demonstrate when the rail must be modeled discretely supported and a nonlinear contact model between rail and wheel must be used, for a correct description of railway noise generation.

Track properties and train speed influence the contact
force between the rail and the wheel as well as the vibrational behavior of the rail. Three cases are studied: Track #1 is for a Korean slab-track. Track #2 is for a Korean ballast track. Parameters for the tracks are listed in Table 1. Euler beam theory is used with a reduction in the moment of inertia to 75% of its nominal value, thus extending the useful frequency region for vertical rail vibrations up to and above 2 kHz [11]. For the Track #1, sleeper mass $m_s$ and ballast stiffness $K_b$ are assumed to be infinite in the numerical simulations. The combined roughness of the wheel and the rail used to generate line texture heights $y_s$ is shown as one third octave band wavenumber spectra in Fig. 3. The roughness levels of the wheel are relatively low and originate from measurements performed on wheels from the trailer coaches of Swedish X2-train presented in [13]. In addition, the roughness levels of the rail are low because the Swedish intercity track section from which it originated was relatively new ground when it was measured. The rail roughness measurements were performed as part of the EU project QCity and are presented in [14]. The combined roughness levels generated from the measurements are extended to cover longer wavelengths, the dashed part of the roughness spectra in Fig. 5, which extends the investigation to lower excitation frequencies.

The above presented measured roughness does not cover the range of short wavelengths required to perform detailed contact modeling with the aim of computing the state-dependent parameters. Instead, a set of roughness data of the wheel and the rail in short wavelengths from [15] is used as reference since the values correspond well with the roughness levels and the decaying slopes presented in [13,14] in a region of wavelengths where the two data sets overlap. Using the short wavelength roughness from [15] together with a method to generate surface data from line texture data (see [12]), surface roughness is generated and superimposed on the geometrical shape of the wheel and the rail.

### 2.2 Rail Green’s function

Track parameters according to Table 1 are used to calculate the rail receptance (Fig. 4). The rail is excited in the middle of a sleeper span and on a support position. The receptance curve for Track #1 has two maxima/peaks. The first curve at low frequencies was determined by the rail mass per unit length $m_r$ and pad stiffness $K_p$ (Table 1). The second curve is the pinned-pinned frequency, where a half bending wavelength equals the distance between the support points. For Track #2 and Track #3, another maxima/resonance occurs when rail and sleeper masses vibrate out-of-phase with the stiffness of the rail pad in between, around 500 Hz for Track #2, and around 300 Hz for Track #3. The location of this resonance depends on the pad stiffness, which is higher for Track #2 than for Track #3.

The vibration propagations of the rail along the rail-sleeper systems are depicted in Fig. 5. The pinned-pinned frequency propagates freely, while vibrations at, for example, 200 Hz decay rapidly. A periodic system such as a periodically supported rail exhibits stop- and pass-band properties, which determine the decay rate of vibrations at different frequencies. An advantage of the current model is that the decay rate or propagation coefficients are automatically included in Green’s function [8,9]. Thus, the decay does not have to be measured because as soon as the rail

---

**Table 1. Parameters of three kinds of tracks**

<table>
<thead>
<tr>
<th>Track#1</th>
<th>Track#2</th>
<th>Track#3</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1E11</td>
<td>2.1E11</td>
<td>2.1E11</td>
<td>N/m²</td>
<td>Rail modulus of elasticity</td>
</tr>
<tr>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td></td>
<td>Rail loss factor</td>
</tr>
<tr>
<td>23E-6</td>
<td>23E-6</td>
<td>23E-6</td>
<td>m⁴</td>
<td>Rail moment of area inertia</td>
</tr>
<tr>
<td>60.3</td>
<td>60.3</td>
<td>60.3</td>
<td>Kg/m</td>
<td>Rail mass per unit length</td>
</tr>
<tr>
<td>140</td>
<td>125</td>
<td></td>
<td>kg</td>
<td>Sleeper mass(half)</td>
</tr>
<tr>
<td>0.635</td>
<td>0.654</td>
<td>0.65</td>
<td>M</td>
<td>Sleeper spacing</td>
</tr>
<tr>
<td>110</td>
<td>275</td>
<td>65</td>
<td>MN/m</td>
<td>Pad stiffness</td>
</tr>
<tr>
<td>0.3</td>
<td>0.14</td>
<td></td>
<td></td>
<td>Pad loss factor</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>0.5</td>
<td>MN/m</td>
<td>Ballast stiffness (under a half sleeper)</td>
</tr>
</tbody>
</table>

(※ Track #1 : Korean slab-track, Track #2 : Korean ballast track, Track #3 : “normal” Swedish ballast track)
parameters, such as pad stiffness and sleeper spacing are known, the decay along the rail is also known.

The energy spectrum densities of the vibrations of the rail are shown in Fig. 6. The rail is excited at $x_0$ with a unit point force. The vibration energy of the rail is calculated for three cases, namely the force point in the middle of a span, the force point on a support position, and the mean for all the different force positions in a sleeper span. Radiation of the vertical rail vibrations depends on the integral of Green’s function, $\int_{-\infty}^{\infty} |G(x|x_0)|^2 dx$, according to eq. (2). Fig. 6 shows the energy spectrum density (ESD) of unit point force excited rails for the three cases: the force point

---

**Fig. 4** Rail receptance; Track #1(a), Track#2(b), Track#3(c)

**Fig. 5** Propagation of vertical rail vibrations away from the excitation point in the middle of a sleeper span: Track #1(a), Track#2(b), Track#3(c)
in the middle of a span, the force point on a support position, and mean for all different force positions in a sleeper span.

It is interesting to note the resemblance with the corresponding receptance curves. For noise radiation, the averaged curves (Fig. 7) should be used because all parts of the rail in a sleeper span contribute. Track #2 seems to radiate more below the pinned-pinned frequency and less above compared to the other two tracks with softer rail pads.

Fig. 8 gives an example of the measured and calculated vertical mobilities of a Korean slab track. The mobility of the rail was calculated using the discrete-supported Timoshenko beam model. The calculated vertical mobility of the slab track at mid-span is compared with the measured values. Compared to the results in Fig. 4(a), the frequency area of the first peak corresponds well but the second peak occurred at a lower frequency, about 900 Hz.

2.3 Contact force calculation

The contact force was calculated with three different contact models: state-dependent, Hertzian, and linear.
Investigation of the Dynamic Properties of Railway Tracks using a Model for Calculation of Generation of Wheel/Rail Noise

The combined rail/wheel roughness levels according to section 2.1 were used. Wheel velocity \( v = 56 \text{ m/s (200 km/h)} \), wheel mass \( M_w = 600 \text{ kg} \), and wheel preload \( P = 67.5 \text{ kN} \). The calculation was done with the periodically supported rail model (sec. 1.2.1), but also with a continuously supported rail model (see [11]), i.e., the rail did not have discrete sleeper supports in this case; instead, the mass and stiffness of the supports was “smeared out” under the rail. Results are shown in Fig. 9 and Fig. 10 below. In Fig. 10 the green curve indicates the result with the Hertzian contact model and for a periodically supported rail. The red curve resulted from the state-dependent contact model and was for a periodically supported rail. For the black curve a linear contact model and periodically supported rail were used. The pink curve resulted from the state-dependent contact model and was for a continuously supported rail.

At low frequencies, excitation at the sleeper-passing frequencies plus harmonics dominated the force spectrum; the continuously supported rail model failed to describe this mechanism. Around and above the pinned-pinned frequency, the choice of contact model was crucial. The linear model (with a constant contact stiffness) under predicted the contact force by 5-10 dB as compared to the state-dependent model.

3. Summary

The primary aim of this study was to investigate different calculation models for the rail vibration and the contact force of the wheel and rail. For three kinds of track systems the calculation were performed and the quantitative discrepancies due to the different ways of the calculation methods were observed. For comparison identical roughness data was used for all the calculations of the three types of tracks. In our next study, the roughness data will be used for each type of track individually and the noise radiation from the rail due to the wheel/rail contact and rolling will be calculated and verified.

Acknowledgement

This study was carried out within the framework of the research projects ‘Development of the noise source model and the core technology of the source contribution identification of 400km/h high speed train’ and ‘Railway noise/vibration reduction technology development’ funded by the Korea Agency for Infrastructure Technology Advancement. The authors gratefully acknowledge the support for this research.

References

14. J. Nielsen, O. Lundberg, N. Renard. In-field measurements of the influence of combined wheel and rail damping on railway noise, Project deliverable 5.9, QCity (EU project within the sixth framework programme), 2009.
17. H.Koh et al, Development of the noise source model and the core technology of the source contribution identification of 400km/h high speed train’, Research report 2013